Lecture 15 **First Order Linear Differential Equations**

A **first order differential equation** is one with the general form:

\[
\frac{dy}{dx} + P(x) \cdot y = Q(x)
\]

For instance,

\[
\frac{dy}{dx} + \frac{y}{x} = e^{-x}
\]

This equation cannot be solved in its present form. Then multiplying both sides by \(x\), we get

\[
x \frac{dy}{dx} + y = xe^{-x}
\]

\[
\Rightarrow x \frac{d}{dx} (y) + y \frac{dx}{dx} = xe^{-x}
\]

\[
\Rightarrow \frac{d}{dx} (xy) = xe^{-x}
\]

\[
\Rightarrow \int \frac{d}{dx} (xy) \, dx = \int xe^{-x} \, dx
\]

\[
\Rightarrow xy = \int xe^{-x} \, dx
\]

\[
\Rightarrow xy = -(x + 1)e^{-x} + C \quad \text{using by parts integration}
\]

We can solve the result for \(y\), and get:

\[
y = \frac{1}{x} \{-(x + 1)e^{-x} + C\}
\]

**Solution of a First Order Linear Differential Equation by Definition**

A solution for the differential equation

\[
\frac{dy}{dx} + P(x) \cdot y = Q(x)
\]

Can be determined by:

\[
y = \frac{1}{I(x)} \left[ \int I(x) Q(x) \, dx + C \right]
\]

Where \(I(x) = e^{\int P(x) \, dx}\)

**Class Practice Assignment:**

Page 632 – 633 of the text book: Q.no. 11, 12, 15, 16, 27, 28 and 29
Example 1: Find the general solution for first-order differential equation:

\[ \frac{dy}{dx} - 2xy = x \]

Here \( P(x) = -2x \) and \( Q(x) = x \), then

\[ I(x) = e^{\int P(x)dx} = e^{\int (-2x)dx} = e^{-x^2} \]

Therefore,

\[ y = \frac{1}{I(x)} \left[ \int I(x)Q(x) \, dx + C \right] = \frac{1}{e^{-x^2}} \left[ \int e^{-x^2} x \, dx + C \right] \]

Solving \( \int e^{-x^2} \, dx \) by substitution by supposing \( u = -x^2 \Rightarrow du = -2xdx \), we get

\[ \int e^{-x^2} x \, dx = \frac{-1}{2} \int e^u \, du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-x^2} \]

Thus, \( y = \frac{1}{e^{-x^2}} \left[ -\frac{1}{2} e^{-x^2} + C \right] \)

Example 2: (pg. 632 Ex. 8.2 Q.no. 6) Find the general solution for first-order differential equation:

\[ x \frac{dy}{dx} + 2y = xe^{x^3} \]

Rewriting the differential equation by multiplying both sides by \( x \), we get

\[ \frac{dy}{dx} + 2 \frac{y}{x} = e^{x^3} \]

Here \( P(x) = \frac{2}{x} \) and \( Q(x) = e^{x^3} \), then

\[ I(x) = e^{\int P(x)dx} = e^{\int (2/x)dx} = e^{2\ln x} = e^\ln x^2 = x^2 \]

Therefore,

\[ y = \frac{1}{I(x)} \left[ \int I(x)Q(x) \, dx + C \right] = \frac{1}{x^2} \left[ \int x^2 e^{x^3} \, dx + C \right] \]

Solving \( \int x^2 e^{x^3} \, dx \) by substituting \( u = x^3 \Rightarrow du = 3x^2\, dx \), we get

\[ \int x^2 e^{x^3} \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u = \frac{1}{3} e^{x^3} \]

Thus, \( y = \frac{1}{e^{-x^2}} \left[ \frac{1}{3} e^{x^3} + C \right] \)

Example 3: (pg. 633 Ex. 8.2 Q.no. 19) Find the particular solution for first-order differential equation that satisfies the given condition \( y = -2 \) when \( x = 1 \)

\[ \frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2} \]

Here \( P(x) = \frac{1}{x} \) and \( Q(x) = \frac{1}{x^2} \), then

\[ I(x) = e^{\int P(x)dx} = e^{\int (1/x)dx} = e^\ln x = x \]

Therefore,
\[
y = \frac{1}{I(x)} \left[ \int I(x)Q(x) \, dx + C \right] = \frac{1}{x} \left[ \int x \left( \frac{1}{x^2} \right) \, dx + C \right]
\]
\[
= \frac{1}{x} \left[ \int \frac{1}{x} \, dx + C \right]
\]
\[
= \frac{1}{x} \ln x + C
\]

Thus, \( y = \frac{1}{x} \ln x + C \)

Now using the given condition that \( y = -2 \) when \( x = 1 \), we get

\[
-2 = \frac{1}{1} \ln 1 + C
\]
\[
\Rightarrow C = -2
\]

Hence, \( y = \frac{1}{x} \ln x - 2 \)

Example 4: (pg. 627 e.g. 8.2.3) David deposits $20,000 into an account in which interest accumulates at the rate of 5% per year, compounded continuously. He plans to withdraw $3,000 per year.

a. Set up and solve a differential equation to determine the value \( Q(t) \) of his account \( t \) years after the initial deposit.

b. How long does it take for his account to be exhausted?

Solution:

a. If no withdrawals are made, the value of the account would change at a percentage rate equal to the annual interest rate; that is,

\[
\frac{100Q'(t)}{Q(t)} = 5
\]

Or equivalently, \( Q'(t) = 0.05Q(t) \). This is the rate at which interest is added to the account, and by subtracting the annual withdrawal rate of $3,000, we obtain the net rate of change of the account; that is,

\[
\frac{dQ}{dt} = 0.05Q - 3000
\]

net rate of change of \( Q = \) rate at which interest is added − rate at which money is withdrawn

Rewriting this equation as

\[
\frac{dQ}{dt} - 0.05Q = -3000
\]

We recognize it as a first-order linear differential equation with \( p(t) = -0.05 \) and \( q(t) = -3000 \) that we wish to solve subject to the initial condition that \( Q(0) = 20,000 \). The integrating factor for this equation is

\[
I(t) = e^{\int -0.05 \, dt} = e^{-0.05t}
\]

So the general solution is

\[
Q(t) = \frac{1}{e^{-0.05t}} \left[ \int e^{-0.05t}(-3000) \, dt + C \right] = e^{0.05t} \left[ -3000 \frac{e^{-0.05t}}{-0.05} + C \right] = 60,000 + Ce^{0.05t}
\]
Since $Q(0) = 20,000$, we have

$$Q(t) = 60,000 + Ce^{0.05t}$$
$$\Rightarrow 20,000 = 60,000 + Ce^0$$
$$\Rightarrow C = -40,000$$

And therefore,

$$Q(t) = 60,000 - 40,000e^{0.05t}$$

b. The account becomes exhausted when $Q(t) = 0$. Solving the equation

$$Q(t) = 60,000 - 40,000e^{0.05t}$$
$$\Rightarrow 0 = 60,000 - 40,000e^{0.05t}$$
$$\Rightarrow 40,000e^{0.05t} = 60,000$$
$$\Rightarrow e^{0.05t} = \frac{60,000}{40,000} = 1.5$$
$$\Rightarrow 0.05t = \ln 1.5$$
$$\Rightarrow t = 8.11$$

Thus, the account is exhausted in approximately 8 years.