Lecture 17 **Power Series**

A series of the form:

\[ a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n \]

is called a **power series** in the variable \( x \).

It may **converge** if \( |x| < R \) i.e. \(-R < x < R\)

**Power Series Representation of Functions**

We can have a representation of function as a power series by manipulating the geometric series.

In order to have a form of power series, we need to have the following form of the geometric series:

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{where} \ |x| < 1 \]

This formula has come from:

\[ 1 + x + x^2 + x^3 + \cdots \]

Which is a form of a geometric series with a common ratio \( x \) and first term as 1. Then the sum of this series will be:

\[ \frac{a}{1-r} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \]

Where the series will converge if \( |x| < 1 \) or \(-1 < x < 1\).

**Differentiation and Integration of Power Series**

If the power series converges for \( |x| < R \), then we observe that the given function is may be a derivative of some function, therefore, can use that function to manipulate by power series.

If \( f \) be the function defined as

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]

Then the differentiation of this function can be:

\[ f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \]

And the integral of this function can be:

\[ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C \]

**Class Practice Assignment:**

Page 727 – 728 of the text book, Ex. 9.3: Q.no. 9 – 14, 24

**Example 1:** Find a power series for a given function and determine its interval of absolute convergence.

a. \( \frac{1}{1+x} \)

b. \( \frac{1}{1-x^3} \)

c. \( \frac{1}{1+9x^2} \)

d. \( \frac{1}{2+3x} \)
Solutions:

a. \( \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \) for \(-1 < x < 1\)

b. \( \frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n} \) for \(-1 < x < 1\)

c. \( \frac{1}{1+9x^2} = \sum_{n=0}^{\infty} (-1)^n (3)^{2n} x^{2n} \) for \(-\frac{1}{3} < x < \frac{1}{3}\)

d. \( \frac{1}{2+3x} = \sum_{n=0}^{\infty} (-1)^n x^n \) for \(-\frac{2}{3} < x < \frac{2}{3}\)

e. \( \frac{x}{4x+1} = \sum_{n=0}^{\infty} (-1)^n (2)^{2n} x^{n+1} \) for \(-\frac{1}{4} < x < \frac{1}{4}\)

f. \( \frac{x}{9+x^2} = \sum_{n=0}^{\infty} (-1)^n (2^n-1) x^{2n+1} \) for \(-3 < x < 3\)

Example 2: (pg. 718 e.g. 9.3.4) Find a power series for a given function \( \ln(1 + x) \)

Solution:

We know that first order derivative of \( f(x) = \ln(1 + x) \) can be given as \( f'(x) = \frac{1}{1+x} \).

Using \( f'(x) \), we get the power series of it as:

\[
f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \quad \text{where } |x| < 1
\]

Now we can use the above result to find the power series for \( f(x) \):

\[
f(x) = \int f'(x) \, dx = \int \sum_{n=0}^{\infty} (-x)^n \, dx = \sum_{n=0}^{\infty} \int (-x)^n \, dx = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}(-1)
\]

Example 3: Find a power series for a given function:

\[
\frac{x^2}{(1-2x)^2}
\]

Solution:

\[
\frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-2x)^2}
\]

The key observation here is

\[
\frac{d}{dx} \left( \frac{1}{1-2x} \right) = \frac{2}{(1-2x)^2}
\]

Now let \( (x) = \frac{1}{1-2x} \), then rewriting it in form of power series, we get

\[
\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n
\]
Taking the derivative with respect to $x$ on both sides, we get

$$\frac{d}{dx} \left( \frac{1}{1-2x} \right) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} 2^n x^n \right)$$

$$\Rightarrow \frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} \quad \text{(note that the } n \text{ will start from 1 instead of 0)}$$

$$\Rightarrow \frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n \quad \text{(if } n \text{ starts from 0 then } n \text{ will be replaced by } n+1)$$

Thus, the given function can be manipulated by power series as:

$$\frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-2x)^2}$$

$$= x^2 \cdot \frac{2}{(1-2x)^2}$$

$$= \frac{x^2}{2} \cdot \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} 2^n (n+1) x^{n+2}$$